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# PROOF OF SPRINGER'S HYPOTHESIS

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ABSTRACT

Let $G$ be a reductive group over a finite field $k$ of a characteristic $p$. $\Pi: G_k \to \mathrm{Aut}\, U$ is an irreducible representation of $G$ in "a general position". Springer formulated a conjecture about values of the character of $\Pi$ on unipotent elements. This conjecture is proved in the article.

The present paper is devoted to the description of a class of complex representations of reductive groups over finite fields. The question of the description of representations of such groups was treated already in the nineteenth century. Namely, Frobenius described all characters of irreducible representations of $SL(2, \mathbf{F}_q)$, the group of $2 \times 2$ matrices of determinant 1 over $\mathbf{F}_q$. Then in the beginning of the 1950's the irreducible characters of $GL(3, \mathbf{F}_q)$ and $GL(4, \mathbf{F}_q)$ were described and in 1955 Green described all irreducible characters of $GL(n, \mathbf{F}_q)$, any $n$. It turned out that the values of those characters on semisimple elements can be easily written down, but the most complicated task is to give their values for unipotent elements. The functions which give those values (for $GL(n, \mathbf{F}_q)$) are Green polynomials. Moreover the larger part of representations was easily partitioned into series (to every conjugacy class of tori there corresponded a series) and values of characters of one series on unipotent elements did not depend on a particular choice of a representation inside this series.

In 1968 Srinivasan described irreducible characters of $Sp(4, \mathbf{F}_q)$. It turned out that also in this case the larger part of representations falls in series, each series corresponds to a conjugacy class of tori, and for representations of one series the restriction of their characters to unipotent elements does not depend on the choice of character.

In consequence the hypothesis was formulated that for any connected reductive group $G$ over finite field $\mathbf{F}_q$ and for any maximal torus $T$ of $G$, there exists a series of irreducible representations of $G$ parameterized by $\hat{T}$, and that

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the restriction of all characters of a fixed series to unipotent elements coincides and is given by a function $Q_G(T,\ )$ which depends only on the torus $T$.

In 1971, T. A. Springer [5] proposed a formula for the function $Q_G(T,u)$, $u$ varying over the set $V$ of unipotent elements of $G$. He proved recently that for nonconjugate tori $T$ and $T'$, the restrictions of $Q_G(T,\ )$ and $Q_G(T',\ )$ to $V$ are orthogonal and that $\Sigma_{v\in V} \mid Q_G(T,v)\mid^2 = \mid V\mid\cdot\mid W_T\mid$, where $W_T$ is the Weyl group of $T$ [6].

In the present paper we show that Springer's hypothesis is correct in the case when the characteristic $p$ of $\mathbf{F}_q$ is big enough. Namely, for any character $\theta$ of torus $T$ we construct a continuation $Q_G(T;\theta)$ of $Q_G(T)$ to the whole group $G$ and prove that $Q_G(T;\theta)$ is a virtual character of $G$. It follows from Springer's results on the functions $Q_G(T)$ that $\Sigma_{g\in G}\mid Q_G(T;\theta;g)\mid^2 = \mid G\mid\cdot\mid W_T(\theta)\mid$, where $W_T(\theta)$ is the stabilizer of $\theta$ in $W_T$. Hence $Q_G(T;\theta)$ is an irreducible character for $\theta$ in general position (that is, when $W_T(\theta)=1$).

Quite recently, Deligne and Lusztig constructed for any character $\theta$ of a maximal torus $T$ of $G$ the action of $G$ on an algebraic variety $X_T$ furnished with a locally constant étale sheaf $\mathcal{F}_\theta$. (Their construction is a generalization of Drinfelds' work, done for SL(2).) This data gives rise to the representation of $G$ in the Euler characteristic of $\mathcal{F}_\theta$, which is denoted $R^\theta_T$. Deligne and Lusztig [2] have shown that

$$\sum_{g\in G} \mid \operatorname{Tr} R^\theta_T(g)\mid^2 = \mid G\mid\cdot\mid W_T(\theta)\mid.$$

Our results imply immediately that

$$\operatorname{Tr} R^\theta_T(g) = \pm Q_G(T;\theta;g).$$

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Let $\boldsymbol{G}$ be a connected reductive Lie group over $k=\mathbf{F}_q$, $G$ be the set of its $k$-points, $\mathfrak{g}$ be the Lie algebra of $\boldsymbol{G}$ and $\mathrm{g}$ be the set of its $k$-points.

Let us denote by $\boldsymbol{V}\subset\boldsymbol{G}$ the variety of unipotent elements in $\boldsymbol{G}$ and by $\mathfrak{w}\subset\mathfrak{g}$ the variety of nilpotent elements in $\mathfrak{g}$. Let further $\boldsymbol{U}\subset\boldsymbol{G}$ be a maximal unipotent $k$-subgroup in $\boldsymbol{G}$ and $\mathfrak{u}\subset\mathfrak{g}$ be its Lie algebra.

The following assumption is made throughout.

Assumption (*). The maps $\ln: \boldsymbol{V}\to\mathfrak{w}$ and $\exp: \mathfrak{w}\to\boldsymbol{V}$ are well defined and the Campbell–Hausdorff formula holds for them.

Under this assumption the Killing form of $\mathfrak{g}$ and its restrictions to all proper reductive subalgebras are nondegenerate.

Denote by $\mathfrak{u}'$ the space of linear functionals on $\mathfrak{u}$. The group $U$ acts naturally upon $\mathfrak{u}$ and $\mathfrak{u}'$. For $\lambda \in \mathfrak{u}'$ let us denote by $B_\lambda(\ ,\ )$ the bilinear alternate form on $\mathfrak{u}$ given by

$$B_\lambda(u_1, u_2) = \lambda[u_1, u_2].$$

LEMMA 1. *There exists a subalgebra $h^\lambda \subset \mathfrak{u}$ which is a maximal isotropic subspace for $B_\lambda$. Any such subalgebra is called subordinate to $\lambda$.*

PROOF. Let $\mathfrak{u} = \mathfrak{u}_0 \supset \mathfrak{u}_1 \supset \cdots \supset \mathfrak{u}_N = 0$ be a series of normal subalgebras such that $\dim \mathfrak{u}_i/\mathfrak{u}_{i+1} = 1$ for $0 \leqq i \leqq N-1$. Denote by $\boldsymbol{h}_i^\lambda \subset \mathfrak{u}_i$ the null-subspace of the restriction of $B_\lambda$ to $\mathfrak{u}_i$ and set $\boldsymbol{h}^\lambda = \bigcup_{i=0} \boldsymbol{h}_i^\lambda$. $\boldsymbol{h}^\lambda$ is subordinate to $\lambda$.

Let $\boldsymbol{H}^\lambda = \exp \boldsymbol{h}^\lambda$ and denote by $\Lambda$ the function on $\boldsymbol{H}^\lambda$ given by $\Lambda(h) = \lambda(\ln h), h \in \boldsymbol{H}^\lambda$. We shall call $\boldsymbol{H}^\lambda$ a subordinate subgroup to $\lambda$. According to Assumption (*), $\boldsymbol{H}^\lambda$ is a subgroup of $\boldsymbol{U}$ and $\Lambda: \boldsymbol{H}^\lambda \to \bar{k}_+$ is a homomorphism into the additive subgroup of $k$. Choose a nontrivial additive character $\psi: k_+ \to \boldsymbol{C}^*$ of $k$. Set $\phi_\lambda = \psi \circ \Lambda: H^\lambda \to \boldsymbol{C}^*$. Then $\varphi_\lambda$ is a character of $H^\lambda$. Consider the representation $\Pi_\lambda$ of $U$ given by $\Pi_\lambda = \mathrm{Ind}_{H^\lambda}^{u}(\varphi_\lambda)$. Let $\chi_\lambda(u) = \det \mathrm{Tr}(\Pi_\lambda(u))$, $u \in U$.

PROPOSITION 1. $$\chi_\lambda(u) = q^{\frac{-\dim \boldsymbol{\Omega}_\lambda}{2}} \cdot \sum_{\mu \in \Omega_\lambda} \psi(\mu(\ln u)).$$

PROOF. Let $\chi_\lambda(w) = \chi_\lambda(\exp w)$, $w \in \mathfrak{u}$. By definition $\chi_\lambda(w) = [H^\lambda]^{-1} \cdot \Sigma_{u \in U}\, a_\lambda(w^u)$, where $a_\lambda = \psi(\lambda(w))$ for $w \in h^\lambda$ and $a_\lambda(w) = 0$ otherwise. Denote by $L^\lambda$ the affine subspace of $\mathfrak{u}'$ consisting of those functionals $\nu/h^\lambda = \lambda$. It is clear that

$$a_\lambda(w) = [L^\lambda]^{-1} \sum_{\nu \in L^\lambda} \psi(\nu(w)).$$

Therefore

$$\chi_\lambda(w) = \frac{1}{[H^\lambda][L^\lambda]} \sum_{\substack{u \in U \\ \nu \in L^\lambda}} \psi(\nu(w^u))$$

$$= \frac{1}{[H^\lambda][L^\lambda]} \sum_{\substack{u \in U \\ \nu \in L^\lambda}} \psi(\nu^u(w)).$$

LEMMA 2. *The group $\boldsymbol{H}_\lambda$ preserves $\boldsymbol{L}^\lambda$ and acts transitively on it.*

PROOF. The first assertion is evident. Denote by $h_0^\lambda \subset \mathfrak{u}$ the null-subspace of $B_\lambda$ and by $\boldsymbol{H}_0^\lambda \subset U$ the stabilizer of $\lambda$. By (*), $\boldsymbol{h}_0^\lambda = \ln(\boldsymbol{H}_0^\lambda)$. Let us consider the map $H_0^\lambda \backslash \boldsymbol{H}^\lambda \xrightarrow{\sigma} \boldsymbol{L}_\lambda$ given by $\sigma(h) = \lambda^h$.

It follows from $h_0^\lambda = \ln \boldsymbol{H}_0^\lambda$ that $\sigma$ is injective. Since $\boldsymbol{H}_0^\lambda \backslash \boldsymbol{H}_0^\lambda \sim \boldsymbol{L}^\lambda$ is the affine space of dimension $\dim \boldsymbol{\Omega}_\lambda/2$, the lemma is proved.

COROLLARY. *$H_\lambda$ acts transitively on $L_\lambda$.*

PROOF. Since Galois cohomology with coefficients in a connected unipotent group is trivial, our corollary follows by standard arguments from Lemma 2.

We can now rewrite the formula for $\chi_\lambda$:

$$\chi_\lambda(w) = \frac{1}{[L^\lambda]} \sum_{\nu \in \Omega_\lambda} \psi(\nu(w)) = q^{\frac{-\dim \Omega_\lambda}{2}} \cdot \sum_{\nu \in \Omega_\lambda} \psi(\nu(w)).$$

Proposition 1 is proved.

PROPOSITION 2. a) *The representation $\Pi_\lambda$ does not depend on the choice of $\boldsymbol{h}^\lambda$.*

b) *$\Pi_\lambda$ is irreducible.*

c) *The representations $\Pi_\lambda$ and $\Pi_{\lambda'}$ are equivalent iff $\Omega_\lambda = \Omega_{\lambda'}$.*

d) $\dim \Pi_\lambda = [\Omega_\lambda]^{1/2}$.

e) *Any irreducible representation of $U$ is equivalent to one of $\Pi_\lambda$, $\lambda \in \mathfrak{u}'$.*

PROOF. a) Since any representation of a finite group is completely determined by its character, our assertion follows immediately from Proposition 1.

b) To prove b), we should check the equality

$$\frac{1}{[U]} \sum_{u \in U} |\chi_\lambda(u)|^2 = 1.$$

This later equality also follows immediately from Proposition 1.

c) Proved by the same argument as a).

d) Evident.

e) It follows from d) that $\Sigma_{\lambda \in \mathfrak{u}_0'} (\dim \Pi_\lambda)^2 = [\mathfrak{u}'] = [\mathfrak{u}] = [U]$, where the sum is taken over the subset $\mathfrak{u}_0' \subset \mathfrak{u}'$ of representatives of orbits. Since representations $\Pi_\lambda$ are irreducible and pairwise inequivalent, e) is proved.

Let $A$ be a regular element in $\mathfrak{g}$. For $x \in V$ set

$$Q_G(x, A) = [U]^{-1} \sum_{y \in \Omega(A)} \psi(B(x, y)),$$

where $B$ is the Killing form of $\mathfrak{g}$ and $\boldsymbol{\Omega}(\boldsymbol{A})$ is the orbit of $A \in \mathfrak{g}$ under the adjoint representation of $\boldsymbol{G}$ in $\mathfrak{g}$. Since the centralizer of a semisimple regular element in a reductive group is connected, $\Omega(A)$ is the orbit of $A$ under $G$.

THEOREM 1. *The restriction of $Q_G(\ln x, A)$ to $U$ is a character of some representation of $U$.*

PROOF. It is sufficient to check that for any irreducible representation $\Pi$ of $U$ the scalar product $\langle Q_G(\ ,A), \operatorname{Tr}\Pi\rangle_U$ $\operatorname{Tr}\Pi\rangle_U$ is a natural number of zero. According to Proposition 2 we can take $\Pi = \Pi_\lambda$ for some $\lambda \in \mathfrak{u}'$. Let $h^\lambda \subset \mathfrak{u}$ be a subordinate subalgebra for $\lambda$, $H^\lambda = \exp h^\lambda$ and $\Lambda$ be a corresponding character of $H^\lambda$. By definition $\Pi_\lambda = \operatorname{Ind}_{H^\lambda}^U(\Lambda)$. By Frobenius reciprocity one has

$$\langle Q_G(\ ,A), \operatorname{Tr}\Pi_\lambda\rangle = \langle Q_G(\ ,A), \Lambda\rangle_{H^\lambda} = [H^\lambda]^{-1}[U]^{-1} \sum_{h\in H^\lambda} Q_G(\ln h, A)\cdot \Lambda^{-1}(h)$$

$$= [H^\lambda]^{-1}[U]^{-1} \sum_{f\in h^\lambda} Q_G(f,A)\psi(-\lambda(f))$$

$$= [H^\lambda]^{-1}[U]^{-1} \sum_{y\in\Omega(A)} \sum_{f\in h^\lambda} \psi[B(f,y)-\lambda(f)].$$

Since $h^\lambda$ is a linear subspace and since $B(\ ,y)-\lambda(\ )$ is linear for any $y \in \mathfrak{g}$, we have

$$\langle Q_G(\ ,A), \operatorname{Tr}\Pi\rangle = \frac{1}{[U]}[X^\lambda].$$

Here $\boldsymbol{X}^\lambda \subset \boldsymbol{\Omega}(\boldsymbol{A})$ is an algebraic variety consisting of those $y \in \boldsymbol{\Omega}(\boldsymbol{A})$ for which the restriction of $B(\ ,y)$ to $h^\lambda$ coincides with $\lambda$.

Since $\lambda$ is assumed fixed we shall omit index $\lambda$ in our notation.

To prove that $[U]$ divides $[X]$ we shall use the following general result whose proof was given to us by P. Deligne.

PROPOSITION 3 (P. Deligne). *Let $\boldsymbol{Z}$ be an algebraic variety defined over a finite field k. Denote by $\tilde{Z}$ the corresponding variety over the algebraic closure $\bar{k}$ of k. Suppose that there exists a partition of $\tilde{Z}$ into a disjoined union of finite number of constructive sets: $\tilde{Z} = \bigcup_1^k \tilde{Z}_i$ such that for every $k, 1 \leqq k \leqq N$ $\tilde{Z}_k$ is open in $\bigcup_{i\geqq k}\tilde{Z}_i$ and suppose further that for every i there exist morphisms $f_i : \tilde{Z}_i \to \tilde{Y}_i$ of $\tilde{Z}_i$ to algebraic varieties $\tilde{Y}_i$ such that for any i and any $y \in \tilde{Y}_i$ the fiber $f_i^{-1}(y)$ is either empty or isomorphic to the fixed affine space $\mathbf{A}^n$. Then $[Z]\cdot[k]^{-n}$ is an integer.*

The proof is in the Appendix.

It follows now from Proposition 3 that to prove Theorem 1 it is sufficient to check that $\boldsymbol{X}$ satisfies the assumptions of Proposition 3 with $n = \dim \boldsymbol{U}$.

To begin with let us describe the partition of $\tilde{X}$. Since we are working now over an algebraically closed field, the regular element $A$ is split and belongs to a split torus $\mathcal{T}$ of Lie algebra $\mathfrak{g}$. We can assume that this torus $\mathcal{T}$ corresponds to a torus $T$ in $\boldsymbol{G}$, which normalizes $U$. Let $W$ be the Weyl group of $G$. For $w \in W$

let us denote by $U_w^-$ (resp. $U_w^+$) the subgroup of $U$ generated by the root subgroups $U_\alpha$ such that $\alpha$ is positive and $\alpha^w$ is negative (resp. $\alpha^w$ is positive). It is well known that for every $w \in W$, $U_w^+ U_w^- = U_w^- U_w^+ = U$. By Bruhat decomposition the group $G$ is the finite disjoint union $G = \bigcup_{w \in W} TU_w^- wU$ of constructive subvarieties $G_w = TU_w^- wU$. For $w \in W$ set $\Omega_w(A) = \mathrm{Ad}\, G_w(A)$ and $X_w = X \cap \Omega_w(A)$. By definition $\Omega_w = \mathrm{Ad}(U)\,\mathrm{Ad}\, w\, \mathrm{Ad}\, U_w^-(A)$. Since $A$ is regular in $\mathcal{T}$ we have $\mathrm{Ad}\, U_w^-(A) = A + \mathfrak{u}_w^-$, where $\mathfrak{u}_w^-$ is the Lie algebra of $U_w^-$. Hence $\Omega_w(A) = \mathrm{Ad}\, U(A^w + \mathrm{Ad}\, w(\mathfrak{u}_w^-))$. Since for all $w \in U$, $f \in \mathcal{T}$, $u \in \mathfrak{u}$, one has $B(\mathrm{Ad}\, u(f), u) = 0$, it follows that $X_w$ is isomorphic to the set of pairs ($u \in U$, $w \in \mathfrak{u}_w^-$) such that for every $f \in h$ one has $B(\mathrm{Ad}\, u\, \mathrm{Ad}\, w(w), f) = \lambda(f)$.

Let us remark now that the set of functionals on $\mathfrak{u}$ of the form $\mathrm{Ad}\, w(u)$, $w \in \mathfrak{u}_w^-$, coincides (taking into account that $\mathfrak{u} = \mathfrak{u}_w^+ \oplus \mathfrak{u}_w^-$) with the set of those functionals $\mu$ on $\mathfrak{u}$ whose restriction to $(\mathrm{Ad}\, w)(\mathfrak{u}_w^+) \subset \mathfrak{u}$ is zero.

Therefore Theorem 1 follows directly from a general Proposition below. To state it we need some notation. Let $U$ be a connected unipotent reduced Lie group over $k$ with Lie algebra $\mathfrak{u}$, $L, H \subset U$ be connected subgroups with Lie algebras $\mathcal{L}$ and $h$, $\lambda$ be a linear functional on $h$ fixed by $\mathrm{Ad}\, H$. Let us denote by $\mathfrak{u}'$ the space of functionals on $\mathfrak{u}$ and by $X \subset \mathfrak{u}' \times U$ the subvariety of pairs $(\mu, u)$ such that $\mu|_{\mathcal{L}} = 0$ and $\mathrm{Ad}\, u(\mu)|_h = \lambda$.

PROPOSITION *The variety $X$ satisfies assumptions of Proposition 3 with $n = \dim U$.*

PROOF. Consider the action of $L \times H$ on $U$ given by

$$(l, h)(u) = l^{-1}uh.$$

It follows from Rosenlicht's Theorem that there exists a finite $L \times H$ invariant partition $U = \cup U_i$ into constructive subsets $U_i$ such that the quotient space $\varphi_i : U_i \to \mathcal{L}_i$ of $U_i$ with respect to the action of $L \times H$ exists. Take $X_i = X \cap (\mathfrak{u}' \times U_i)$ and define morphism $f_i : X_i \to Y_i$ by $f : (w', u) = \varphi_i(u)$.

Let us now compute the fibers of $f_i$. Denote by $\mathcal{L}^\perp \subset \mathfrak{u}'$ the subspace of functionals whose restrictions to $\mathcal{L}$ are zeros and by $h_\lambda^\perp \subset \mathfrak{u}'$ the affine subspace of functionals which are equal to $\lambda$ on $h$. Let $l = \dim L$, $h = \dim H$. Then $\dim \mathcal{L}^\perp = n - l$ and $\dim h_\lambda^\perp = n - h$. For $y \in Y_i$ let us denote by $\phi_y$ the fiber $\varphi_i^{-1}(y) \subset U_i$. Let $u \in \phi_y$. It is clear that $\phi_y$ is isomorphic to the quotient space $uLu^{-1} \cap H \backslash L \times H$. Since $L$ and $H$ are connected unipotent groups, the quotient space $\phi_y$ is isomorphic to an affine space of dimension $l + h - m$, where $m = \dim uLu^{-1} \cap H$, and projection $\pi : L \times H \to \phi_y$ admits a regular section $\gamma : \phi_y \to L \times H$. Set $\psi = \mathrm{Ad}\, u(\mathcal{L}^\perp) \cap h_\lambda^\perp$. It is clear that either $\psi = \varnothing$ or $\psi$ is isomorphic to intersection $\operatorname{Ad} u(\mathcal{L}^{\perp}) \cap h^{\perp}$ and hence is an affine space of dimension

$$(n-l)+(n-h)+m-n=n+m-l-h.$$

The map $\alpha: \psi \times \phi_y \to \mathfrak{u}' \times U$ given by $\alpha(\psi, \varphi)=(\gamma(\varphi)\psi, \varphi)$ gives rise to an isomorphism $\alpha: \psi \times \phi \tilde{\to} f^{-1}(Y_1)$. Theorem 1 is proved.

REMARK. T. A. Springer, in one of his talks at the time of his visit to Moscow, informed us that he can prove that the function $Q_G(A, x)$ depends (for unipotent $x \in V \subset G$) only on the centralizer of regular element $A$, that is, it depends only on the maximal torus containing it. The discussion which follows will not use this result. However, it permits us to simplify the notation and therefore we shall assume the above result of T. A. Springer and write $Q_G(T, x)$ in place of $Q_G(A, x)$. See the proof in [6].

As before, let $G$ be a connected reductive $k$-group, $T \subset G$ be a maximal $k$-torus and $\theta: T \to \mathbf{C}^*$ be a character. Using these data, we shall now define the class function $Q_G(\theta; T; g)$ on $G$. Our definition is inductive. We shall assume that $Q_H$ is defined for all proper reductive subgroups $H \subset G$. For $g \in G$ let $g = su$ be its Jordan splitting, $s$ semisimple, $u$ nilpotent and $s$ commutes with $u$.

I. If $s$ is not conjugate in $G$ to an element of $T$, then we set $Q_G(\theta; T; g) = 0$.

II. If $s$ is conjugate in $G$ to an element of $T$, we assume (as we can) that $s \in T$.

a) If $s \in C(G)$, we set $Q_G(\theta; T; g) = \theta(s) Q_G(T; u)$.

b) If $s \notin C(G)$, we set

$$Q_G(\theta; T; g) = (-1)^{\alpha(g)} |Z^0(s)|^{-1} \sum_{\substack{x \in G \\ x^{-1}sx \in T}} Q_{Z^0(s)}(xTx^{-1}, u)\, \theta(x^{-1}sx).$$

Here $Z^0(s)$ is the connected component of the centralizer $Z(s)$ of $s$ in $G$, $\alpha = \sigma(G) - \sigma(Z^0(s))$, $\sigma$ is the split semisimple rank.

MAIN THEOREM. *For any pair $(\theta, I)$ the class function $Q_G(\theta; T;\ )$ is a virtual character of $G$.*

The proof of this result is rather long. To begin it we shall prove several simple lemmas about structures. Let $\mathscr{C} = (C_i)$ be a partially ordered set with maximal element $C_0$.

DEFINITION. The Möbius function is a function $\mu$ on $\mathscr{C}$ such that $\mu(C_0) = 1$ and for any $C \in \mathscr{C}$, $C \neq C_0$, $\Sigma_{C' \in \mathscr{C}, C' \geqq C}\, \mu(C') = 0$

It is clear that $\mu$ is uniquely determined by these properties and that its range are integers.

Let us say that $\mathscr{C}$ is a set with intersections if for any pair $C_1, C_2 \in \mathscr{C}$ the partially ordered set $\mathscr{C}(C_1, C_2) = \{C \in \mathscr{C} \mid C \leqq C_1, C \leqq C_2\}$ contains a maximal element. It will be denoted $C_1 \cap C_2$.

LEMMA 3. *Let $\mathscr{C}$ be a finite partially ordered set with intersections. Then for any $C \in \mathscr{C}$, $C \neq C_0$ and any $C_1 \leqq C$ one has $\Sigma_{C' \cap C = C_1} \mu(C') = 0$.*

PROOF. Denote by $\mathscr{D} \subset \mathscr{C}$ the subset of $D \in \mathscr{C}$ such that $C_1 \leqq D \leqq C$ and define function $\lambda$ on $\mathscr{D}$ by the system of equations

$$\lambda(C_1) = 1; \forall D \in \mathscr{D} \neq C_1, \sum_{\substack{D' \in \mathscr{D} \\ D' \leqq D}} \lambda(D') = 0.$$

Then for any function $\alpha$ on $\mathscr{C}$ the following equality holds:

$$\sum_{D \in \mathscr{D}} \lambda(D) \sum_{\substack{C' \in \mathscr{C} \\ C' \geqq D}} \alpha(C') \overset{(1)}{=} \sum_{C' \in \mathscr{C}} \alpha(C') \sum_{\substack{D \in \mathscr{D} \\ D \leqq C'}} \lambda(D) \overset{(2)}{=} \sum_{C' \in \mathscr{C}} \alpha(C') \sum_{\substack{D \in \mathscr{D} \\ D \leqq C' \cap C}} \lambda(D) \overset{(3)}{=} \sum_{\substack{C' \in \mathscr{C} \\ C' \cap C = C_1}} \alpha(C')$$

((1) is evident; (2) follows from the definition of intersection; (3) is implied by the fact that for $C' \geqq C$ one has $C' \cap C \in \mathscr{D}$). In particular

$$\sum_{\substack{C' \in \mathscr{C} \\ C' \cap C = C_1}} \mu(C') = \sum_{D \in \mathscr{D}} \lambda(D) \sum_{\substack{C' \in \mathscr{C} \\ C' \geqq D}} \mu(C') = 0$$

since $D \leqq C \neq C_0$. Lemma 3 is proved.

Let again $G$ be a connected reductive group and $\boldsymbol{T} \subset \boldsymbol{G}$ be a maximal $k$-subtorus of $\boldsymbol{G}$.

DEFINITION. Let us say that a connected reductive $k$-subgroup $\boldsymbol{H} \subset \boldsymbol{G}$ is distinguished if there exists a subgroup $\boldsymbol{T}_0 \subset \boldsymbol{T}$ such that $\boldsymbol{H} = Z^0_G(\boldsymbol{T}_0)$. It is clear that $\boldsymbol{H} \supset \boldsymbol{T}$ and $\boldsymbol{H}$ is generated by $\boldsymbol{T}$ and those root subgroups of $\boldsymbol{G}$ with respect to $\boldsymbol{T}$ whose restriction to $\boldsymbol{T}_0$ is 1. Let us denote by $\mathscr{H}$ the set of distinguished subgroups of $\boldsymbol{G}$ partially ordered by inclusion. $\mathscr{H}$ is a set with intersections: if $\boldsymbol{H}_1 = Z^0_G(\boldsymbol{T}_1)$, $\boldsymbol{H}_2 = Z^0_G(\boldsymbol{T}_2)$, then $\boldsymbol{H}_1 \cap \boldsymbol{H}_2 = Z^0_G(\boldsymbol{T}_1 \cdot \boldsymbol{T}_2)$. Denote by $\mu$ the Möbius function of $\mathscr{H}$ and consider the function $K_G(\theta; T;\ )$ on $G$ given by

$$K_G(\theta; T) = \sum_{H \in \mathscr{H}} (-1)^{\sigma(H)} \mu(H) \operatorname{Ind}^G_H Q_H(\theta; T;\ ),$$

where $\mathrm{Ind}_H^G$ is the operation of induction of class functions from $H$ to $G$.

PROPOSITION 4. *The support of the functions $K_G(\theta; T;\ )$ is contained in $C(G)\times V$. (Recall that $C(G)$ is the center of $G$ and $V\subset G$ is the set of unipotent elements.)*

PROOF. Let $g = su$, $s$ is semisimple and $u$ unipotent, $su = us$. Let us denote by $s_\alpha$ ($\alpha\in A$, $A$ a set of indices) the set of elements of $T$ which are conjugate to $s$ in $G$. Let us denote further by $L_\alpha$ the centralizer of $s_\alpha$ in $G$ and by $\Gamma_\alpha^H$ the set of elements of $T$ which are conjugate to $s_\alpha$ in $H$.

Set $I_H = \mathrm{Ind}_H^G Q_H(\theta; T)(g)$. By the definition of induction one has

$$I_H = [H]^{-1}\sum_{x\in G} Q_H(\theta; T; g^x) = [H]^{-1}\sum_{x\in G} Q_H(\theta; T; s^x u^x)$$

(where, as usual, $Q_H(\theta; T; h)$ is taken to be equal to zero if $h\notin H$). If $s^x u^x\in H$ then $s^x\in H$. Hence

$$I_H = [H]^{-1}\sum_{\substack{x\in G\\ s^x\in H}} Q_H(\theta; T; s^x u^x).$$

It follows from the definition of $Q_H(\theta; T)$ that $Q_H(\theta; T; s^x u^x) = 0$ if $s^x$ cannot be conjugate to an element of $T$ by an element of $H$. Therefore

$$\begin{aligned} I_H &= [H]^{-1}\sum_{\alpha\in A} |s_\alpha^H|\cdot|\Gamma_\alpha^H|^{-1}\sum_{\substack{x\in G\\ s^x=s_\alpha}} Q_H(\theta; T; s_\alpha u^x)\\ &= \sum_{\alpha\in A}|\Gamma_\alpha^H|^{-1}\cdot|L_\alpha\cap H|^{-1}\sum_{\substack{x\in G\\ s^x=s_\alpha}} Q_H(\theta; T; s_\alpha u^x).\end{aligned}$$

By definition (where $[M]$ denotes the number of rational points of $M$)

$$\begin{aligned} &\sum_{\substack{x\in G\\ s^x=s_\alpha}} Q_H(\theta; T; s_\alpha u^x)\\ &\quad = (-1)^{\sigma(H)-\sigma((L_\alpha\cap H)^0)}|(L_\alpha\cap H)^0|^{-1}\sum_{\substack{x\in G\\ s^x=s_\alpha}}\ \sum_{\substack{h\in H\\ s_\alpha^h\in T}} Q_{(L_\alpha\cap H)^0}(hTh^{-1}, u^x)\theta(s_\alpha^h)\\ &\quad = (-1)^{\sigma(H)-\sigma((L_\alpha\cap H)^0)}\sum_{s_\beta\in\Gamma_\alpha^H}\frac{|L_\beta\cap H|}{|(L_\beta\cap H)^0|}\sum_{\substack{x\in G\\ s^x=s_\beta}} Q_{(H\cap L_\beta)^0}(T, u^x)\theta(s_\beta).\end{aligned}$$

Therefore

$$I_H = \sum_\alpha \frac{(-1)^{\sigma(H)-\sigma((L_\alpha\cap H)^0)}}{|(L_\alpha\cap H)^0|}\sum_{\substack{x\in G\\ s^x=s_\alpha}} Q_{(H\cap L_\alpha)^0}(T, u^x)\theta(s_\alpha).$$

Hence

$$K_G(\theta, T)(g) = \sum_{H \in \mathcal{H}} (-1)^{\sigma(H)} \mu(H) I_H$$

$$= \sum_{\alpha \in A} \theta(s_\alpha) \sum_{\substack{x \in G \\ s^x = s_\alpha}} \sum_{H \in \mathcal{H}} \mu(H) \frac{(-1)^{\sigma((L_\alpha \cap H)^0)}}{|(L_\alpha \cap H)^0|} Q_{(L_\alpha \cap H)^0}(T, u^x)$$

$$= \sum_{\alpha \in A} \theta(s_\alpha) \sum_{\substack{x \in G \\ s^x = s_\alpha}} \sum_{H' \subset L_\alpha} \frac{(-1)^{\sigma(H')}}{|H'|} Q_{H'}(T, u^x) \sum_{\substack{H \in \mathcal{H} \\ (H \cap L_\alpha)^0 = H'}} \mu(H').$$

If $s \notin C(G)$ then $L_\alpha \neq G$ and $K_G(\theta; T; g) = 0$ by Lemma 3. Proposition 4 is thus proved.

PROPOSITION 5. *For all unipotent elements $u \in G$ the quotient*

$$|Z_G(u)|^{-1} \cdot |C(G)| \cdot K_G(\theta; T; u)$$

*lies in the ring* $\mathbf{Z}[1/q]$.

PROOF. We assume the assertion is proved for all connected reductive subgroups of dimension $< \dim G$.

If $u \in H$, then, by definition,

$$Q_H(T, u) = |U \cap H|^{-1} \sum_{y \in \Omega(A)} \psi(B(y, w)) = |U \cap H|^{-1} |T|^{-1} \sum_{h \in H} \psi(B(A^{h^{-1}}, w))$$

$$= |U \cap H|^{-1} |T|^{-1} \sum_{h \in H} (\psi \circ B)(A, w^h),$$

where $w = \ln u$. Let us denote by $(\psi \circ B)_H(A, \cdot)$ the function on $\mathfrak{g}$ given by

$$(\psi \circ B)_H(A, y) = \begin{cases} \psi(B(A, y)) & \text{for} \quad y \in H \\ 0 & \text{for} \quad y \notin H. \end{cases}$$

Then

$$\mathrm{Ind}_H^G Q_H(T)(u) = |H \cap U|^{-1} |T|^{-1} \sum_{x \in G} (\psi \circ B)_H(A, w^x)$$

$$= \frac{|Z_G(u)|}{|U \cap H|} \sum_{\gamma \in Z_G(u) \backslash G/T} \frac{(\psi \circ B)_H(A, w^\gamma)}{|T \cap Z_G(u^\gamma)|}.$$

Hence

$$K_G(\theta; T; u) = |Z_G(u)| \sum_{H \in \mathcal{H}} (-1)^{\sigma(H)} \frac{\mu(H)}{|U \cap H|} \sum_{\gamma \in Z_G(u) \backslash G/T} \frac{(\psi \circ B)_H(A, w^\gamma)}{|T \cap Z_G(u^\gamma)|}$$

$$= |Z_G(u)| \sum_{\gamma \in Z_G(u) \backslash G/T} |T \cap Z_G(u^\gamma)|^{-1} \times$$

$$\times \sum_{H \in \mathcal{H}} \frac{(-1)^{\sigma(H)}\mu(H)}{|U \cap H|} (\psi \circ B)_H(A, w^\gamma)$$

$$= |Z_G(u)| \sum_{\gamma \in Z_G(u)\backslash G/T} |T \cap Z_G(u^\gamma)|^{-1} \sum_{\substack{H \in \mathcal{H} \\ H \ni u^\gamma}} \frac{(-1)^{\sigma(H)}\mu(H)}{|U \cap H|} (\psi \circ B)(A, w^\gamma).$$

Let us set $H_\gamma = Z_G^0(T \cap Z_G(u^\gamma))$. Since condition $u^\gamma \in H$ is equivalent to $H \supset H_\gamma$, we have

$$K_G(\theta; T; u) \cdot |Z_G(u)|^{-1} = \sum_{\gamma \in Z_G(u)\backslash G/T} \frac{|C(G)|}{|C(H_\gamma)|} (\psi \circ B)\ (A, w^\gamma)$$
$$\times \sum_{\substack{H \in \mathcal{H} \\ H \supset H_\gamma}} \frac{(-1)^{\sigma(H)}\mu(H)}{|U \cap H|}.$$

To conclude the proof of Proposition 5 we shall prove

LEMMA 4. *Let $H_0 \subset G$ be a distinguished subgroup and $T_0 \subset T$ be its center. Then*

$$|C(G)| \sum_{H > H_0} |H \cap U|^{-1}(-1)^{\sigma(H)}\mu(H)$$

*is divisible by $|T_0|$ in the ring $\mathbf{Z}[q^{-1}]$.*

The proof of Lemma 4 is rather long and will be given in a number of steps. The main step is Proposition 6 having independent interest.

Let $\mathcal{P} = \{\mathcal{P}_i\}$ be the set of classes of $k$-parabolic subgroups in $G$ and let us fix a representative $P_i \in \mathcal{P}_i$. For any $k$-parabolic subgroup $P$ of $G$ let $s(P)$ denote its corank, that is, the split semisimple rank of its Levy subgroup. Set $X_i = P_i \backslash G$ and denote by $X_i' \subset X$ the subvariety of those points $x \in X_i$ whose stabilizer in $T_0$ is the center $C(G)$ of $G$.

LEMMA 5. *Let $x \in X_i$ and let $T_x$ be the stabilizer of $x$ in $T$. Then $T_x$ is the center of some distinguished subgroup.*

PROOF. Let $P_x \subset G$ be the stabilizer of $x$ in $G$. Then $P_x$ is a parabolic subgroup and there exists a maximal torus $T'$ of $G$, which contains $T_x$ and is contained in $P_x$. Consider the centralizer $H_x = Z_G(T_x)$ of $T_x$ in $G$. Then $H_x$ contains $T$ and $T'$. Hence $C(H_x) \subset T \cap T' = T_x$ whence the assertion.

PROPOSITION 6. *The following equality holds:*

$$\sum_{H > H_0} \mu(H)(-1)^{\sigma(H)} \cdot |U \cap H| = \sum_i (-1)^{s(P_i)} |X_i'|.$$

PROOF. For every distinguished subgroup $\boldsymbol{H} > \boldsymbol{H}_0$ let us denote by $\boldsymbol{T}_H \subset \boldsymbol{T}$ its center and by $\boldsymbol{X}_{i,H} \subset \boldsymbol{X}_i$ the subvariety of fixed points of $\boldsymbol{T}_H$. Lemma 5 implies immediately the equality $|\boldsymbol{X}'_i| = \Sigma_{H > H_0} \mu(H) | X_{i,H} |$. Hence

$$\sum_i (-1)^{s(P_i)} | X'_i | = \sum_{H > H_0} \mu(H) \sum_i (-1)^{s(P_i)} | X_{i,H} |.$$

To conclude the proof of our Proposition it is sufficient to check the validity of the following equality:

$$\sum_i (-1)^{s(P_i)} | X_{i,H} | = (-1)^{\sigma(H)} \cdot | U \cap H |.$$

In the case when $T_H$ contains an element $t$ such that $Z_G(t) = \boldsymbol{H}$, our equality turns (by Lemma 5) into the well-known formula for the character of the Steinberg representation of the $\boldsymbol{G}$. In the general case it should follow along the same lines and would result in Proposition 6.

Let us denote by $\boldsymbol{T}_0^0$ the connected component of $\boldsymbol{T}_0$ and by $N$ the quotient $N = \boldsymbol{T}_0/\boldsymbol{T}_0^0$. It is known that there exists a polynomial $a(t) \in \mathbf{Z}[t]$ such that $| T_0^0(\mathbf{F}_{q^r}) | = a(q^r)$. Let us denote by $N(r)$ the order of $N(\mathbf{F}_{q^r})$. Analogously let us denote by $C(t)$ such a polynomial that $C(q^r) = | \boldsymbol{C}^0(G)(\mathbf{F}_{q^r}) |$ and also let us set $N'(r) = \boldsymbol{C}(G)/\boldsymbol{C}^0(G)(\mathbf{F}_{q^r})$. It follows from Lang's Theorem (cf. [4]) that $| T_0(\mathbf{F}_{q^r}) | = N(r) \cdot a(q^r)$. Let us put further $I(H) = \dim(\boldsymbol{U} \cap \boldsymbol{H})$ and

$$P_{H_0}(t) = \sum_{H > H_0} (-1)^{\sigma(H)} \mu(H) t^{(H)}.$$

LEMMA 6. *The quotient* $b(t) \stackrel{\text{def}}{=} P_{H_0}(t) \cdot C(t)/a(t)$ *is a polynomial. Moreover* $b(q^r)$ *is an integer and* $N(r)/N'(r)$ *divides* $b(q^r)$.

PROOF. Proposition 6 implies that $P_{H_0}(g^r) = \Sigma_i (-1)^{s(P_i)} | X'_i(\mathbf{F}_{g^r}) |$. Since $\boldsymbol{T}_0$ acts on $\boldsymbol{X}'_i$ fixed-point-free, it follows that $| X'_i(\mathbf{F}_{q^r}) |$ is divisible by $| T_0(\mathbf{F}_{q^r}) | = N(r) a(q^r)$. Therefore the rational function $b(t)$ takes integer values at points $t = q^r$. Hence $b(t)$ is a polynomial, as asserted.

Let us go now to the study of $N(r)$. The Frobenius automorphism $F$ acts on $\bar{N} = N(\bar{\mathbf{F}}_q)$ and $N(r)$ is the number of fixed points of $F^r$ in $\bar{N}$. Since $\bar{N}$ is a finite group there exists a natural number $n$ such that $F^n$ acts trivially on $\bar{N}$. Hence the number of fixed points of $F^{nr-1}$ is the number of fixed points of $F^{-1}$, which is equal to the number of fixed points of $F$. Hence $N(nr - 1) = N(1)$. Analogously $N'(rn - 1) = N'(1)$.

We return now to the proof of Proposition 5. We should verify divisibility of $P_{H_0}(q^{-1})$ by $| T_0 | = N(1) a(q)$. Since $a(q^{-1}) = \pm a(q) q^{-\dim T_0}$, $C(q^{-1}) =$

$\pm C(q)\cdot q^{-\dim C(G)}$, it is sufficient to check that the value of $N(1)^{-1}\cdot N'(1)\cdot b(t)$ in $t=q^{-1}$ falls into $\mathbf{Z}[q^{-1}]$. Note first that the coefficients of $N(1)^{-1}\cdot N'(1)\cdot b(t)$ are rational. Let $d$ be the least common multiple of their denominators. Represent $d$ in the form $d=p^m\cdot d'$, where $(p,d')=1$, $p=\operatorname{char} k$. Let $l$ be a natural number such that $q^{ln}-1$ is divisible by $d'$. Then

$$N'(1)b(q^{ln-1})/N(1)=N'(ln-1)b(q^{ln-1})/N(ln-1)$$

is an integer (by Lemma 6). Therefore

$$N'(1)b(q^{-1})/N(1)=N'(1)b(q^{ln-1})/N(1)+N'(1)[b(q^{-1})-b(q^{ln}\cdot q^{-1})]/N(1).$$

Now the choice of $l$ guarantees that $N'(1)\cdot b(q^{-1})/N(1)\in\mathbf{Z}[q^{-1}]$. Thus Lemma 4 and, therefore, Proposition 5 are proved.

To complete the proof of the Main Theorem we shall use the following easy result:

LEMMA 7. *Let $M$ be a $p$-group, $f(m)$, $m\in M$, be a character of $M$ and $l$ be an integer relatively prime to $p$. If values of the class function $l^{-1}\cdot f(m)$ are algebraic integers then $l^{-1}f$ is a character of $M$.*

The proof follows immediately from the fact that the index of the subgroup of virtual characters in the group of integral class functions is of the form $p^N$.

PROOF OF THE MAIN THEOREM. Let us consider the function $K_G(\theta,T)$ on $G$. By the induction hypothesis and by Theorem 1 the restriction of this function to $U$ is a virtual character. By Proposition 4 the support of $K_G(\theta,T)$ consists of unipotent elements. We shall now prove that $K_G(\theta,T)$ is a virtual character of $G$. By Brauer's Theorem it is sufficient to prove that the restriction of $K_G(\theta,T)$ to subgroups of $G$ of the form $G_s\times G_u$, where $G_s$ consists of semisimple elements and $G_u$ consists of unipotent elements, is a virtual character.

Since the support of the restriction of $K_G(\theta,T)$ to $G_s\times G_u$ is contained in $G_u$, that restriction is equal to $\operatorname{Ind}_{G_u}^{G_s\times G_u}([G_s]^{-1}\cdot K_G(\theta,T))\big|_{G_u}$. By Proposition 5 the restriction of $[G_s]^{-1}\cdot K_G(\theta,T)$ to $G_u$ is integer-valued. Since $|G_s|$ is relatively prime to $|G_u|$, it follows from Lemma 9 that this restriction is a virtual character. Hence $K_G(\theta,T)$ is a virtual character of $G$.

It follows now from the definition of $K_G(\theta,T)$ and from the induction assumption that $Q_G(\theta,T)$ is a virtual character of $G$. The Main Theorem is proved.

THEOREM 3. *$Q_G(\theta,T)$ coincides with the function $(-1)^{\sigma(G)-\sigma(T)}R^{\theta}_{T,G}$ introduced by Deligne and Lusztig* (cf. [2]).

PROOF. We shall assume that the theorem is proved for all groups of dimension $< \dim G$. Then it follows immediately that $Q_G(\theta, T) - (-1)^{\sigma(G)-\sigma(T)} R^{\theta}_{T,G}$ is supported by the set of unipotent elements. It follows from results of T. A. Springer and Deligne–Lusztig (cf. [2] and [6]) that

$$\Sigma \mid Q_G(\theta, T) - (-1)^{\sigma(G)-\sigma(T)} R^{\theta}_{T,G} \mid^2 \leqq 4 \mid W_T \mid \cdot \mid V \mid,$$

where $V$ is the set of unipotent elements of $G$.

On the other hand, the fact that $Q_G(\theta, T)$ and $R^{\theta}_{T,G}$ are virtual characters of $G$ implies that the sum above should be divisible by $\mid G \mid$. So it is zero, as asserted.

REMARK. I think that Assumption (*) made in the beginning about the existence of ln for unipotent elements is inessential. E. Gutkin proved that any irreducible representation of the maximal unipotent subgroup $U$ of $GL(n, \mathbf{F}_q)$ is induced from the one-dimensional representation of the group of $\mathbf{F}_q$-points of the appropriate connected algebraic subgroup of $U$. If an analogous result will be proved for the maximal unipotent subgroup of an arbitrary reductive group, Assumption (*) would become superfluous, and in the case of good characteristics the proof would need only minor corrections.

## APPENDIX

PROOF OF PROPOSITION 3. We will prove the proposition only in the case when (**) $Z_i$ is open in $\bigcup_{j \geqq i} Z_j$ for all. It is easy to see that it is enough for our purposes.

Let $H^j_c(\tilde{Z}, \mathbf{Q}_e)$ be the étale cohomology of $\tilde{Z}$ with compact supports ([4]), $\sigma: \tilde{Z} \to \tilde{Z}$ the Frobenius morphism over $k$ and $\varphi_j$ the corresponding linear homomorphisms $\varphi_j: H^j_c(\tilde{Z}, Q_e) \to H^j_c(\tilde{Z}, \mathbf{Q}_e)$. It is well known ([1]) that the eigenvalues of $\varphi_j$ are algebraic integers, and $\mathrm{Tr}\, \varphi_j \in \mathbf{Z}$. The Lefschetz formula ([4]) tells us that $[Z] = \Sigma_{j=0}^{2\dim \tilde{Z}} (-1)^j \mathrm{Tr} \varphi_j$. So Proposition 3 immediately follows from

PROPOSITION 3'. *Under the assumption of Proposition* 3 *and* (**) *the eigenvalues of* $\varphi_j$ *are divisible by* $q^n$.

PROOF OF PROPOSITION 3'. Let $K$ be a finite extention of $k$. It is clear that Proposition 3' is true for $Z$ iff it is true for $Z \otimes_{\mathrm{Spec}\, k} \mathrm{Spec}\, K$. So we can suppose that the decomposition $\tilde{Z} = \cup \tilde{Z}_i$ and the morphisms $f_i: \tilde{Z}_i \to \tilde{Y}_i$ are defined over $k$.

Firstly we prove that the eigenvalues of the action $\sigma$ on $H^j_c(\tilde{Z}_i, \mathbf{Q}_e)$ are divisible by $q^n$. For this we consider the Leray spectral sequence corresponding to morphism $f_i : \tilde{Z}_i \to \tilde{Y}_i$.

Then we obtain $E_2^{p,q} \to H_c^{p+q}(\tilde{Z}_i, Q_e)$ where $E_2^{p,q} = H_c^p(Y_i, R_c^q f_i \bar{\mathbf{Q}}_e)$. We know ([3]) that a) $R^q f$ commutes with the base change, and b) $H_c^j(\mathbf{A}^n) = 0$ if $j \neq 2n$ and $H_c^{2n}(\mathbf{A}^n) = \mathbf{Q}_e(n)$. So we see that $H_c^j(\tilde{Z}_i, \mathbf{Q}_e) = H_c^{j-2n}(\tilde{Y}_i, \mathbf{R}^{2n} f_i \mathbf{Q}_e)$ and for every closed point $\bar{y} \in \tilde{Y}_i$ the action of the Frobenius $\sigma_{\bar{y}}$ on the stalk $(\mathbf{R}^{2n} f_i \mathbf{Q}_e)\bar{y}$ ([1]) is divisible by $[k_{\bar{y}}]^n$. It follows now from ([3]) that the eigenvalues of $\varphi^j$ on $H_c^j(\tilde{Z}_i, \mathbf{Q}_e)$ are divisible by $q^n$.

Let $\boldsymbol{X}_i$ be $\bigcup_{j \geq i} \boldsymbol{Y}_j$. It follows from (**) that $\boldsymbol{X}_i$ is closed in $\boldsymbol{Z}$, and $\boldsymbol{X}_1 = \boldsymbol{Z}$. We will prove by induction that the eigenvalues of the action of Frobenius on $H^i(\tilde{X}_i)$ are divisible by $q^n$.

If $i$ is big enough, $\boldsymbol{X}_i = \varnothing$ and there is nothing to prove so suppose that our statement is true for $\boldsymbol{X}_i$ and prove it for $\boldsymbol{X}_{i-1}$. We have an exact sequence ([3]):

$$H_c^j(\tilde{Z}_{i-1}, \mathbf{Q}_e) \to H_c^j(\tilde{X}_{i-1}, \mathbf{Q}_e) \to H_c^j(\tilde{X}_i, \mathbf{Q}_e) \to H^{j+1}(\tilde{Z}_{i-1}, \mathrm{Q}_e)$$

and all maps in it commute with Frobenius. So our statement follows from the fact that Proposition 3′ is true for $\boldsymbol{X}_i$ and $Z_{i-1}$. Proposition 3′ and, as follows, Proposition 3 are proved.

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